Girraween High School

2026 Year 12 HSC Task 1

Mathematics Extension 2

General Instructions

- Reading time: 5 minutes
- Working time: 90 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Answer multiple-choice questions on the multiple-choice answer sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 83

Section I **5 Marks**

- Attempt Questions 1 - 5
- Allow about 10 minutes for this section

Section II **78 Marks**

- Attempt Questions 6 - 11
- Allow about 80 minutes for this section

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Section I

5 marks
Attempt Questions 1 – 5
Allow about 10 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 5

Question 1

$i^{2026} =$

A) i

B) -1

C) $-i$

D) 1

Question 2

A cubic equation with all real coefficients has roots α, β and γ such that $\alpha^2 + \beta^2 + \gamma^2 < 0$. The equation must have:

A) Three real roots

B) Two real roots

C) One real root

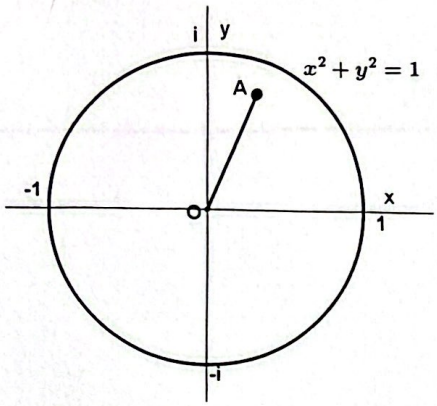
D) No real roots

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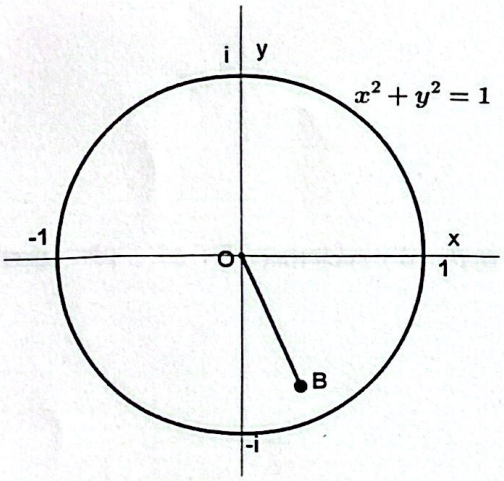
Question 3

z is a complex number in the first quadrant such that $\overrightarrow{OA} = z$ and $|z| < 1$ $(see\ diagram)$.

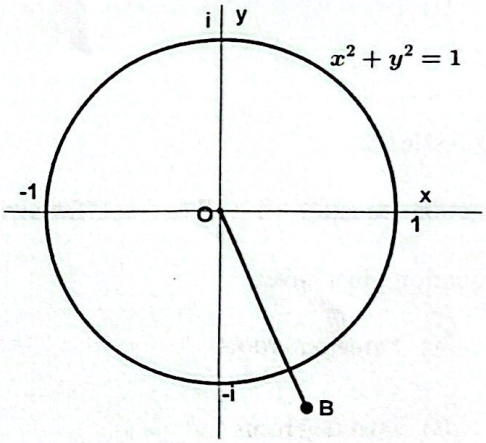


$\overrightarrow{OB} =$ $\frac{1}{z}$ would be represented by the diagram

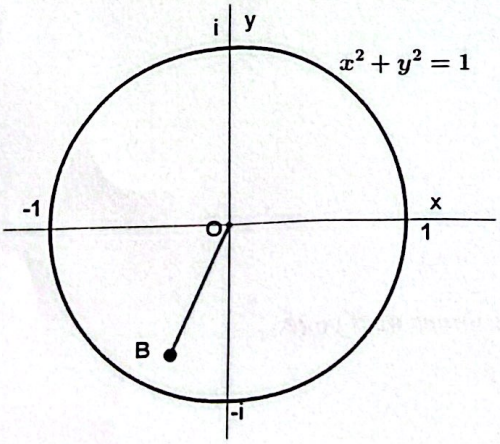
A)



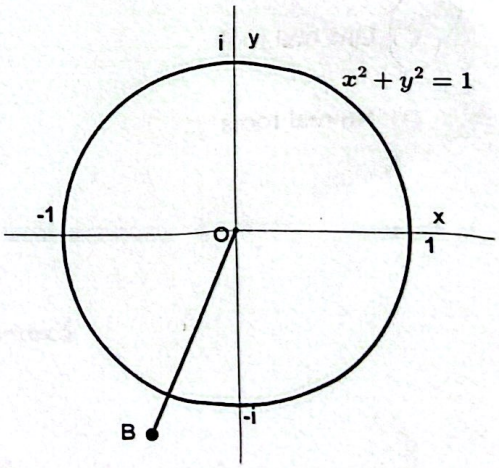
B)



C)



D)



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Question 4

If z is a complex number $x + iy, y \neq 0$ then which of the following is NOT always true?

A) $z + \bar{z}$ is always entirely real

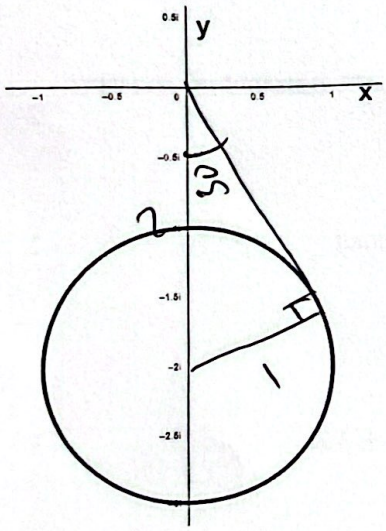
B) $z - \bar{z}$ is always entirely imaginary

C) $z\bar{z}$ is always entirely real

D) $\frac{z}{\bar{z}}$ is always entirely imaginary

Question 5

A complex number z lies on the circle $|z + 2i| = 1$ $(see\ diagram)$



The range of the argument of z is

A) $-\pi \leq Arg\ z \leq 0$

B) $-\frac{3\pi}{4} \leq Arg\ z \leq -\frac{\pi}{4}$

C) $-\frac{5\pi}{6} \leq Arg\ z \leq -\frac{\pi}{6}$

D) $-\frac{2\pi}{3} \leq Arg\ z \leq -\frac{\pi}{3}$

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Section II

78 marks
Attempt Questions 6 - 11
Allow about 80 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

(Note: you may use the notation cis when using modulus/argument form).

Question 6 (12 marks) **Start a new writing page**

a) (i) Express $-1 + i\sqrt{3}$ in modulus/argument form. 2

(ii) Find $\frac{-1+i\sqrt{3}}{-1-i}$ in Cartesian form. 2

(iii) Given $-1 - i = \sqrt{2}(cos - \frac{3\pi}{4} + i\ sin - \frac{3\pi}{4})$ show that the exact value of $sin\ \frac{17\pi}{12} = \frac{-\sqrt{6}-\sqrt{2}}{4}$ 2

b) (i) By letting $(x + iy)^2 = 9 + 40i$, find the two square roots of $9 + 40i$. 3

c)

(ii) Hence solve the equation $z^2 + (3 + 6i)z - (9 + i) = 0$. 3

Examination continues next page…

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Question 7 (11 marks) **Start a new writing page**

a) (i) By completing the square, solve $z^2 - 4z + 8 = 0$ over the complex field. 3

(ii) Show that $z = \frac{2}{3}$ is a double root of the equation 3

$9z^4 + 24z^3 + 28z^2 - 80z + 32 = 0$

(iii) Hence solve $9z^4 + 24z^3 + 28z^2 - 80z + 32 = 0$ over the complex field. 2

b) (i) If $z = -1 + 2i$ is a root of the equation $4z^3 + 5z^2 + 14z - 15 = 0$ 1

explain why $z = -1 - 2i$ is also a root.

(ii) Hence find all of the roots of $4z^3 + 5z^2 + 14z - 15 = 0$. 2

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Question 8 (10 marks) **Start a new writing page**

Sketch the following curves and regions on separate Argand diagrams:

(a) $Re(z) < 1$ and $Im(z) \geq -2$ 3

(b) $|z - 1 - i\sqrt{3}| \leq 2$ and $0 \leq Arg\, z \leq \frac{\pi}{3}$ 3

(c) $|z + 2 - i| = |z - 4 - i|$ 2

$2-(-2+i)$ $\quad$ $2-(4+i)$

(d) $Arg\,(z - 1) = Arg\,(z + i)$ 2

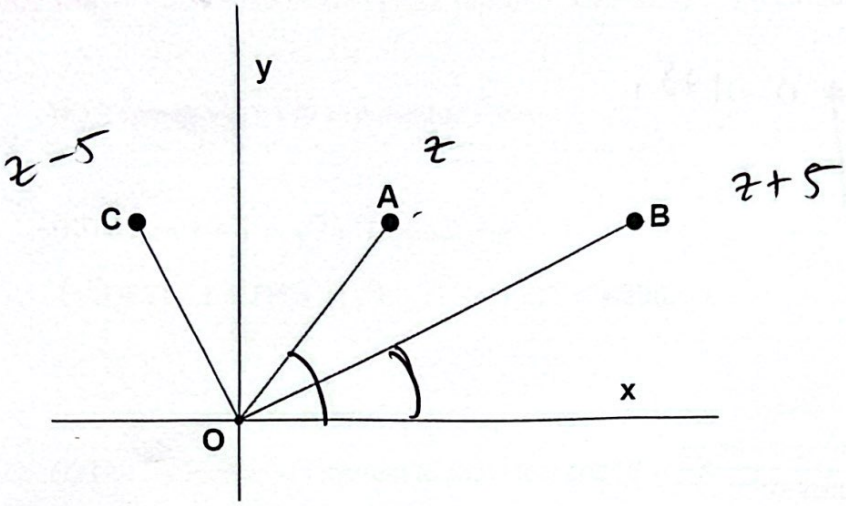
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Question 9 (15 marks) **Start a new writing page**

a) In the diagram below, $\overrightarrow{OA} = z$. $0 < Arg(z) < \frac{\pi}{2}$ and $|z| = 5$.

$\overrightarrow{OB} = z + 5$ and $\overrightarrow{OC} = z - 5$.



(i) Prove $Arg\,(z+5) = \frac{1}{2} Arg\, z$ 2

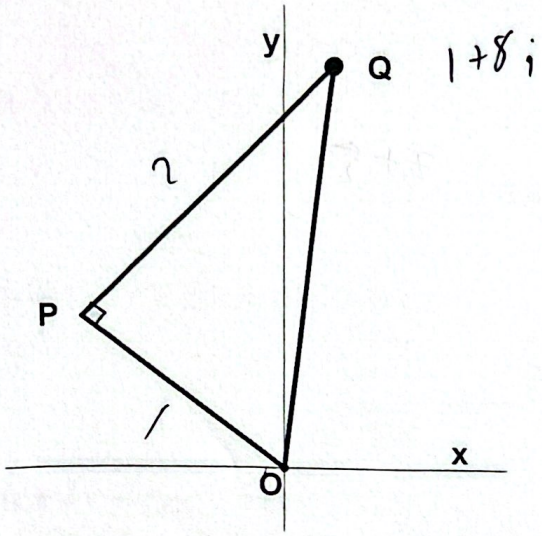
(ii) Prove $Arg\,(z-5) = Arg\,(z+5) + \frac{\pi}{2}$ 2

Question 9 continues next page…

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Question 9 (continued)

b) In the Argand diagram below, $\overrightarrow{OQ} = 1 + 8i$ and triangle OPQ is right angled at P with $PQ = 2 \times OP$ *(see diagram)*. 3



Find the coordinates of P. 2

c) The relationship $Arg\ \frac{z+1}{z-3} = \frac{\pi}{6}$ represents part of a circle in the complex plane.

(i) Find the centre and radius of the circle giving reasons. 4

(ii) Sketch $Arg\ \frac{z+1}{z-3} = \frac{\pi}{6}$ showing all features. 2

Examination continues next page…

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Question 10 (14 marks) **Start a new writing page**

a) Find the five fifth roots of $-32i$. Leave your answers in modulus/argument form. 3

b) (i) Use Euler's theorem to prove that $e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$. 2

(ii) Express $\sqrt{3} + i$ in exponential form. 2

(iii) Given $1 - i = \sqrt{2}e^{-\frac{\pi i}{4}}$, show that 2

$(\sqrt{3} + i)^5(1 - i)^6 + ((\sqrt{3} - i)^5(1 + i)^6 = -256.$

c) (i) Use DeMoivre's Theorem to show that $\cot 4\theta = \frac{cot^4\theta - 6cot^2\theta + 1}{4cot^3\theta - 4cot\theta}$ 3

(ii) Hence show that the solutions to $z^4 + 6z^2 + 1 = 0$ are 2

$\pm i\cot\frac{\pi}{8}$ and $\pm i\cot\frac{3\pi}{8}$.

Examination continues next page…

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Question 11 (16 marks) **Start a new writing page**

a) (i) State the roots of $z^7 - 1 = 0$. You may leave your answers in modulus/argument form. 2

(ii) If w is the root of $z^7 - 1 = 0$ with the smallest positive argument, show that w^n, n a positive integer is also a root of $z^7 - 1 = 0$. 1

(iii) Show that $w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$ 1

(iv) Show that $(w + w^6)(w^2 + w^5)(w^3 + w^4) = 1$ 2

(v) Hence show that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$. 2

(You may assume that if $z = \cos\theta + i\sin\theta$ that $z^n + \frac{1}{z^n} = 2\cos n\theta$).

b) (i) Show that $z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$ 1

(ii) Resolve $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$ into real quadratic factors 2

(you may use your solutions to Question 11(a))

(iii) Hence show that

$$z^3 + \frac{1}{z^3} + z^2 + \frac{1}{z^2} + z + \frac{1}{z} + 1 = \left(z + \frac{1}{z} - 2\cos\frac{2\pi}{7}\right)\left(z + \frac{1}{z} + 2\cos\frac{3\pi}{7}\right)\left(z + \frac{1}{z} + 2\cos\frac{\pi}{7}\right)$$ 2

(iv) Hence by letting $u = z + \frac{1}{z}$, show that $2\cos\frac{2\pi}{7}$, $-2\cos\frac{\pi}{7}$ and $-2\cos\frac{3\pi}{7}$ are the solutions to $u^3 + u^2 - 2u - 1 = 0$. 3

End of Examination.

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2026 Task 1

Mathematics Extension 2

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

correct ↙

A ⊗ B ⊗ C ○ D ○

1.	A	○	B	●	C	○	D	○
2.	A	○	B	○	C	●	D	○
3.	A	○	B	●	C	○	D	○
4.	A	○	B	○	C	○	D	●
5.	A	○	B	○	C	○	D	●

Solutions

Multiple Choice

Q.(1) i^{2026}

$= i^{2024} \times i^2$

$= 1 \times -1$

$= -1.$ (B)

(2) $\alpha^2+\beta^2+\gamma^2<0 \rightarrow$ Must have at least 1 non real root.

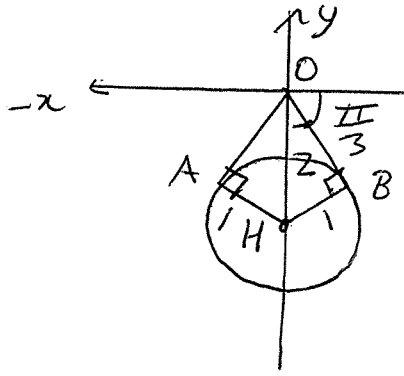
As co-efficients are all real the CONJUGATE of that root is a root.

∴ One real root, 2 non-real = (C)

(3) As $|z|<1, \left|\frac{1}{z}\right|>1$ & $\text{Arg}\left(\frac{1}{z}\right) = -\text{Arg}\, z.$ (B)

(4) (D) is not always true. $\frac{z}{\bar{z}}$ has argument $2\text{Arg}(z)$ which may not be either 0 or π [but could be].

(5)



OB is a tangent to circle centre H.

$\sin \angle HOB = \frac{1}{2}$

$\angle HOB = \frac{\pi}{6}$

$\therefore \angle xOB = \frac{\pi}{3}$

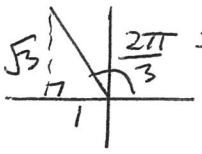
Similarly $\angle -xOA = \frac{\pi}{3}$ (D)

$\therefore -\frac{2\pi}{3} \leq \text{Arg}\, z \leq -\frac{\pi}{3}$

Feedback

Solutions

Q.(6)(a)(i) $-1+i\sqrt{3}$



$= 2\,\text{cis}\,\frac{2\pi}{3}$ Well done.

2

(ii) $\frac{-1+i\sqrt{3}}{-1-i} \times \frac{(-1+i)}{(-1+i)}$

$= \frac{1-i-i\sqrt{3}-\sqrt{3}}{2}$

$= \left(\frac{1-\sqrt{3}}{2}\right) - i\left(\frac{1+\sqrt{3}}{2}\right)$

Well done.

2

(iii) $\frac{-1+i\sqrt{3}}{-1-i}$

$= \frac{2\,\text{cis}\,\frac{2\pi}{3}}{\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)}$

$= \sqrt{2}\,\text{cis}\,\frac{17\pi}{12}$

2

Equating imaginary parts of (ii) & (iii)

$-\frac{1-\sqrt{3}}{2} = \sqrt{2}\sin\frac{17\pi}{12}$

$\therefore \sin\frac{17\pi}{12} = \frac{-1-\sqrt{3}}{2\sqrt{2}} \times\sqrt{2}$

$= -\frac{6-\sqrt{2}}{4}$.

Feedback

Generally well done. Needed to write EQUATING IMAGINARIES.

Solutions Q.(6)(b)(i) $(x+iy)^2 = 9+40i$, x, y real

$(x^2-y^2) + 2ixy = 9+40i$.

Equating reals, $x^2-y^2 = 9$ (1)

Equating imaginaries $2xy = 40$. (2)

Then do either:

Modify (2) & substitute in (1): $y = \frac{20}{x}$ (2)*

$x^2 - \left(\frac{20}{x}\right)^2 = 9$

$\times x^2$ & rearrange

$x^4 - 9x^2 - 400 = 0$

$(x^2-25)(x^2+16) = 0$

$x = \pm 5$ as x real.

As $y = \frac{20}{x}$ from (2)* $y = \pm 4$.

3

OR Method 2

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= 9^2 + 40^2$ [as $(2xy)^2 = 4x^2y^2$]

$x^2+y^2 = \sqrt{9^2+40^2} = 41$.

OR Method 3 MUST JUSTIFY this.

As $z = x+iy = \sqrt{9+40i}$, $|z|^2 = |z^2| = x^2+y^2 = \sqrt{9^2+40^2}^2$ [there is also another way to show $x^2+y^2=41$ which is valid].

$x^2+y^2 = 41$.

& using either Method 2 or 3: $x^2+y^2 = 41$ (1)

$x^2-y^2 = 9$ (2)+

$2x^2 = 32 \Rightarrow x = \pm 4$.

As $y = \frac{20}{x}$, $y = \pm 5$.

Feedback $x+iy = 5+4i$.

As most students used $y = \frac{20}{x}$ to get y the problem of making sure $x+iy = 5+4i$ didn't come up.

NOTE: Some students wrote $y = \frac{20}{\pm 5}$. THIS IS REALLY BAD & SLOPPY MATHS as you don't know which y goes with which x.

Note: If they used $x^2+y^2=41$, $x^2-y^2=9$ then got $x = \pm 5$. if they DIDN'T use $y = \frac{20}{x}$ then they will be penalised as you don't know whether it's $5+4i$ OR $5-4i$. BEST TO USE CONVENTIONAL METHOD

Solutions

Q11 (b) (b) (iii)

$$z = \frac{-(3+6i) \pm \sqrt{(3+6i)^2 - 4 \times 1 \times -(9+i)}}{2}$$

$$= \frac{-(3+6i) \pm \sqrt{27 + 36i + 4(9+i)}}{2}$$

$$= \frac{-(3+6i) \pm \sqrt{9 + 40i}}{2}$$

$$= \frac{-3-6i+(5+4i)}{2} \text{ or } = \frac{-3-6i-5-4i}{2}$$

$z = 1-i$ or $z = -4-5i$.

3

Generally well done.

Feedback

Solutions

Q.(7)(a)(i) $z^2 - 4z + 8 = 0$

$(z-2)^2 + 4 = 0$

$(z-2)^2 = -4$

$z - 2 = \pm 2i$

$z = 2 \pm 2i$

(ii) $P\left(\frac{2}{3}\right) = 9\left(\frac{2}{3}\right)^4 + 24\left(\frac{2}{3}\right)^3 + 28\left(\frac{2}{3}\right)^2 - 80\left(\frac{2}{3}\right) + 32$

$= 0.$

$P'(z) = 36z^3 + 72z^2 + 56z - 80$

$P'\left(\frac{2}{3}\right) = 36\left(\frac{2}{3}\right)^3 + 72\left(\frac{2}{3}\right)^2 + 56\left(\frac{2}{3}\right) - 80$

$= 0$

$z = \frac{2}{3}$ is a double root of $P(z) = 0$

[Note: $P''(z) = 108z^2 + 144z + 56$ & $P''\left(\frac{2}{3}\right) = 200 \neq 0$.]
so DOUBLE ROOT ONLY.

(iii) From (ii) $(3z-2)^2 = 9z^2 - 12z + 4$ is a factor of $P(z)$.

$$
\begin{array}{r}
z^2 + 4z + 8 \\
9z^2 - 12z + 4 \overline{)\,9z^4 + 24z^3 + 28z^2 - 80z + 32} \\
9z^4 - 12z^3 + 4z^2 \\
\hline
36z^3 + 24z^2 - 80z \\
36z^3 - 48z^2 + 16z \\
\hline
72z^2 - 96z + 32 \\
72z^2 - 96z + 32 \\
\hline
0
\end{array}
$$

Feedback

$\therefore P(z) = 0$ becomes $(3z-2)^2(z^2 - 4z + 8) = 0$
& solutions are $z = \frac{2}{3}, \frac{2}{3}, 2 \pm 2i$.

a i) Done well

ii) Done well - not many showed $P''\left(\frac{2}{3}\right) \neq 0$ but were not penalised this time.

(iii) Done well - most used sums & products of roots.

Solutions Q.(7)(b)(i) As $4z^3+5z^2+14z-15=0$ has all real co-efficients & $z=-1+2i$, its conjugate $(z=-1-2i)$ must also be a root (conjugate root theorem).

(ii) By sum of roots: $\alpha+\beta+\gamma = -\frac{b}{a}$

$$(-1+2i)+(-1-2i)+\gamma = -\frac{5}{4}$$

$$-2+\gamma = -\frac{5}{4}$$

$$\gamma = \frac{3}{4}$$

Roots of $4z^3+5z^2+14z-15=0$ are $-1\pm 2i, \frac{3}{4}$.

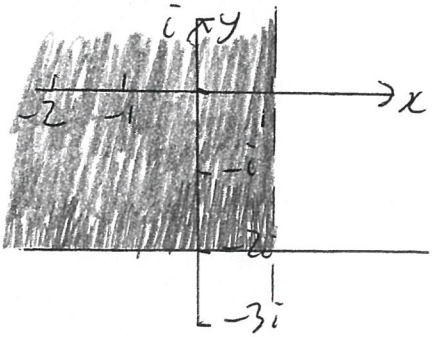
Note: Could also find the third root by dividing $P(z)$ by $[(z+1-2i)(z+1+2i)] = z^2+4z+5$.

Feedback

b i) done well

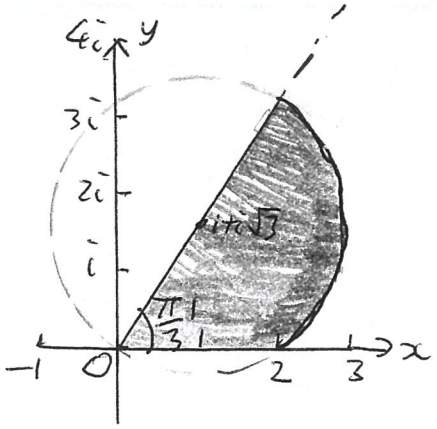
ii) done well

Solutions Q.(8)(a)



(b) $|z-1-i\sqrt{3}| \leq 2$ → Inside circle

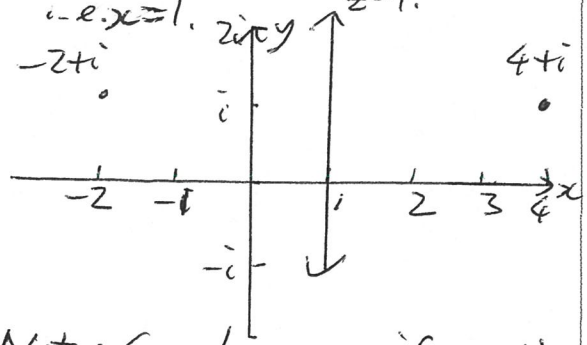
centre $1+i\sqrt{3}$ $r=2$ & $0 \leq \text{Arg } z \leq \frac{\pi}{3}$



(c) $|z+2-i| = |z-4-i|$

⊥ Bisector of line between $-2+i$ & $4+i$

i.e. $x=1$.



Note: Can also use if $z=x+iy$

$\sqrt{(x+2)^2+(y-1)^2} = \sqrt{(x-4)^2+(y-1)^2}$

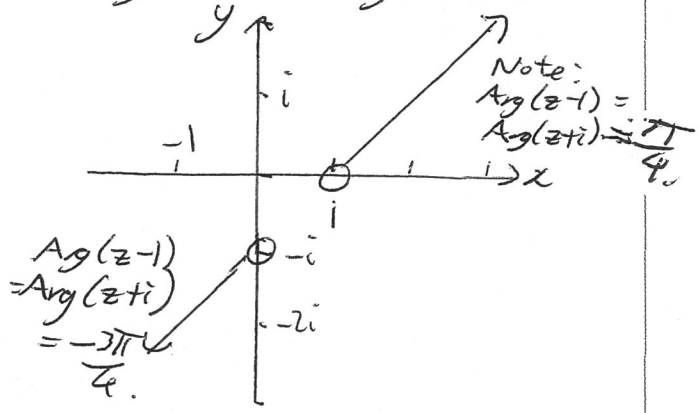
$x^2+4x+4+(y-1)^2 = x^2-8x+16+(y-1)^2$

$-x^2-(y-1)^2+8x-4$

$12x = 12$

$\underline{x=1.}$

(d) $\text{Arg}(z-1) = \text{Arg}(z+i)$

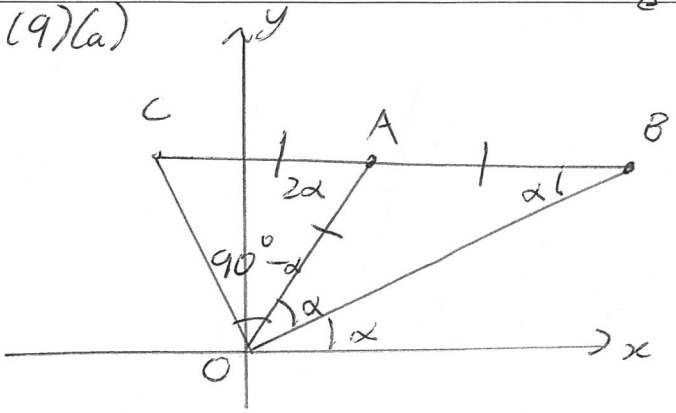


Feedback

a, b, c done well.

(d) Some students got wrong

Solutions Q.(9)(a)



(i) Let $\text{Arg}(z+5) = \alpha$

$OA = AB = 5$

$\angle OBA = \alpha$ [alternate $\angle$'s, $AB // Ox$].

$\angle AOB = \alpha$ [$\angle$'s opposite = sides in isosceles $\triangle$].

$\therefore \text{Arg } z = \angle AOx = 2\alpha = 2 \times \text{Arg}(z+5)$.

(ii) $CA = OA = 5$.

$\angle OAC = 2\alpha$ [exterior $\angle$ $\triangle OAB$].

$\angle COA = \angle OCA = \frac{\pi}{2} - \alpha$ [$\angle$'s opposite = sides in isosceles $\triangle COA$].

$\therefore \text{Arg}(z-5) = \angle COx = \frac{\pi}{2} + \alpha$

$= \frac{\pi}{2} + \text{Arg}(z+5)$.

Feedback

students have given a variety of solutions some of which are included with this

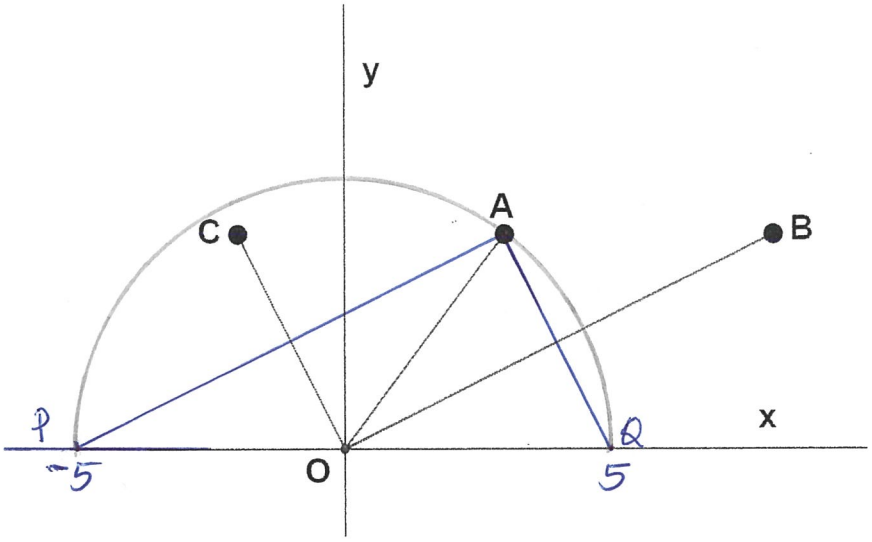
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Alternative Solutions to Q.9

Question 9 (15 marks) **Start a new writing page**

a) In the diagram below, $\overrightarrow{OA} = z$. $0 < Arg(z) < \frac{\pi}{2}$ and $|z| = 5$.

$\overrightarrow{OB} = z + 5$ and $\overrightarrow{OC} = z - 5$.



(i) Prove $Arg\,(z+5) = \frac{1}{2} Arg\, z$ 2

(ii) Prove $Arg\,(z-5) = Arg\,(z+5) + \frac{\pi}{2}$ 2

(i) $\angle APQ = \frac{1}{2} \angle AOQ$ (Angle at the centre is twice angle at the circumference standing on the same arc)

$Arg(z+5) = \frac{1}{2} Arg(z)$

Question 9 continues next page…

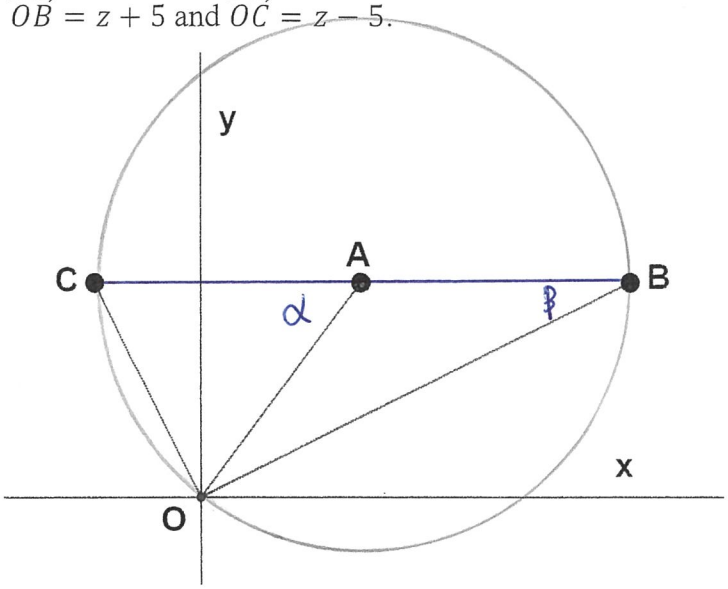
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Alternative Solutions to Q.9

Question 9 (15 marks) **Start a new writing page**

a) In the diagram below, $\overrightarrow{OA} = z$. $0 < Arg(z) < \frac{\pi}{2}$ and $|z| = 5$.

$\overrightarrow{OB} = z + 5$ and $\overrightarrow{OC} = z - 5$.



(i) Prove $Arg\,(z+5) = \frac{1}{2} Arg\, z$ 2

(ii) Prove $Arg\,(z-5) = Arg\,(z+5) + \frac{\pi}{2}$ 2

(i) $\beta = \frac{1}{2}\alpha$ (Angle at the centre is twice angle at the circumference standing on the same arc)

$\alpha = \angle AOX = Arg(z)$ (alternate angles between parallel lines)

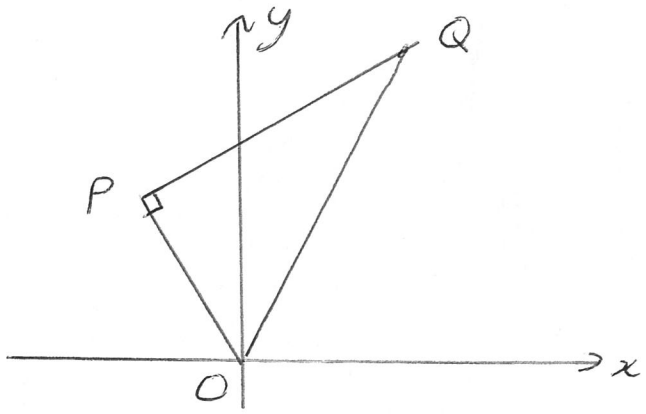
$\beta = \angle BOX = Arg(z+5)$ (alternate angles, $AB \parallel OX$)

$Arg(z+5) = \beta = \frac{1}{2}\alpha$

$= \frac{1}{2} Arg(z)$

Question 9 continues next page…

Solutions Q.(9)(b)



As $PQ = 2 \times OP$ & PQ is at right angles CLOCKWISE from OP,

$$\vec{PQ} = -2i \times \vec{OP}$$

$$\therefore \vec{OQ} = \vec{OP} + \vec{PQ}$$

$$1+8i = \vec{OP} - 2i\vec{OP}$$

$$1+8i = (1-2i)\vec{OP}$$

$$\therefore \vec{OP} = \frac{1+8i}{1-2i} \times \frac{(1+2i)}{(1+2i)}$$

$$= \frac{1+2i+8i-16}{5}$$

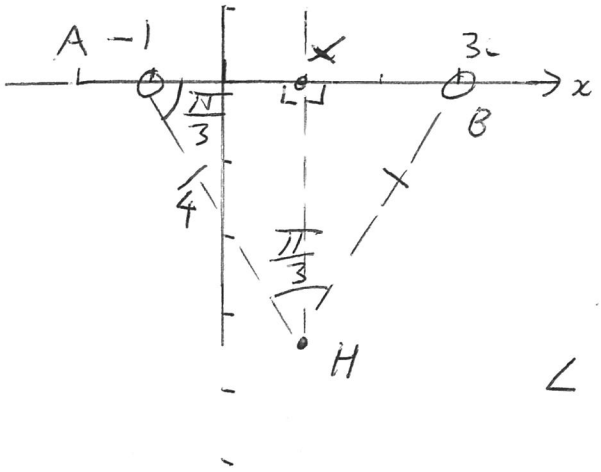
$$= \frac{-5+10i}{5}$$

$$\vec{OP} = -1+2i$$

Feedback

* A hard question for many students.

Solutions Q.(9)(c)(i)



Centre of circle is on $x=1$

Let H = circle centre.

Let $-1=A$, $3=B$

H must be below A & B

as $\text{Arg}(z+1) = \text{Arg}\, AH$

$> \text{Arg}(z-3) = \text{Arg}\, BH$.

$\angle AHB = \frac{\pi}{3}$ [$\angle$ at centre $= 2\times\angle$ at circumference on same arc).

Let $X = 1+0i$.

$AH = BH$ [circle radii).

$\angle HAX = \frac{\pi}{3}$ [$\angle$'s opposite = sides in isosceles $\triangle$ OR equilateral $\triangle$]

$\therefore AH = AB = 4$ [sides in equilateral $\triangle$).

$XH = 4\sin\frac{\pi}{3} = 2\sqrt{3}$.

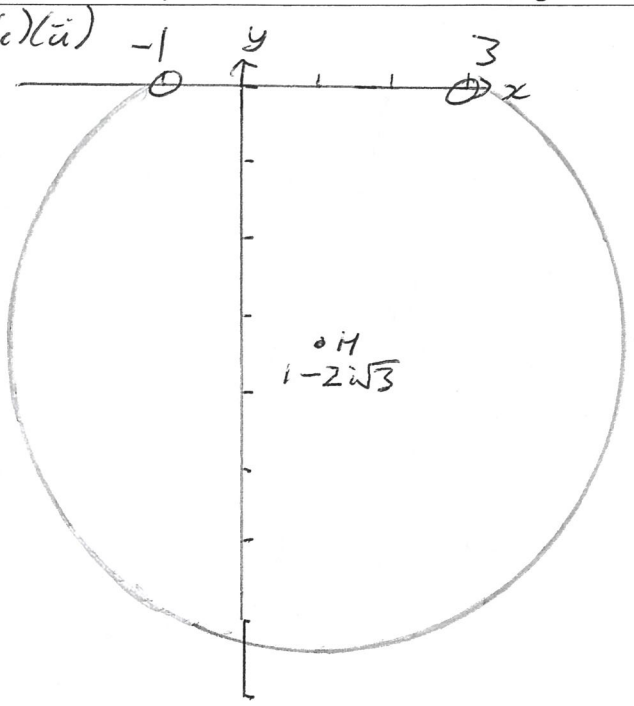
Major arc of circle: Centre $= 1-2i\sqrt{3}$.

Radius $= 4$.

Feedback

Some students got the centre as $1+2i\sqrt{3}$

Solutions (9)(c)(ii)



Feedback

* Some drew the arc above the x axis.
* Forgot to have open circles at -1 and 3.

Solutions

Q.(10)(a) Roots of $-32i$

$= \text{Roots of } 32\,\text{cis}(-\frac{\pi}{2})$

$= 2\,\text{cis}(-\frac{\pi}{10}) + \frac{2k\pi}{5},\ k = 0, 1, 2, 3, 4$

$= 2\,\text{cis}\ \frac{-\pi}{10},\ 2\,\text{cis}\ \frac{3\pi}{10},\ 2\,\text{cis}\ \frac{7\pi}{10},\ 2\,\text{cis}\ \frac{11\pi}{10},\ 2\,\text{cis}\ \frac{3\pi}{2}$

Can also do principal argument $\rightarrow$ $2\,\text{cis}\ \frac{-9\pi}{10},\ 2\,\text{cis}\ \frac{-\pi}{2}$.

(b)(i) By Euler, $e^{ni\theta} = \cos n\theta + i\sin n\theta$

$e^{-ni\theta} = \cos(-n\theta) + i\sin(-n\theta)$

$\therefore e^{ni\theta} + e^{-ni\theta} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$ (as cos even, sin odd).

(ii) $\sqrt{3} + i = 2e^{\frac{\pi}{6}i}$

(iii) Hence, $(\sqrt{3}+i)^5(1-i)^6 + (\sqrt{3}-i)^5(1+i)^6$

$= (2e^{\frac{\pi i}{6}})^5 \times (\sqrt{2}e^{\frac{-\pi i}{4}})^6 + (2e^{\frac{-\pi i}{6}})^5 \times (\sqrt{2}e^{\frac{\pi i}{4}})^6$

$= 32e^{\frac{5\pi i}{6}} \times 8e^{\frac{-3\pi i}{2}} + 32e^{\frac{-5\pi i}{6}} \times 8e^{\frac{3\pi i}{2}}$

$= 256e^{\frac{-2\pi i}{3}} + 256e^{\frac{2\pi i}{3}}$

$= 256 \times 2\cos(\frac{2\pi}{3})$ (from (i)).

$= 256 \times 2 \times -\frac{1}{2}$

$= -256.$

Feedback

10 a) done well

b i) done well

ii) done well

iii) done well

Solutions Q. (10)(c) (i)

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta.$

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ equating reals

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ equating imaginaries.

$\therefore \cot 4\theta = \dfrac{\cos 4\theta}{\sin 4\theta} = \dfrac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$ $\div \sin^4\theta$ $\div \sin^4\theta$

$$= \frac{\dfrac{\cos^4\theta}{\sin^4\theta} - \dfrac{6\cos^2\theta}{\sin^2\theta} + 1}{\dfrac{4\cos^3\theta}{\sin^3\theta} - \dfrac{4\cos\theta}{\sin\theta}}$$

$$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}.$$

(ii) If $\cot 4\theta = 0$, $4\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{3\pi}{8}$.

Hence if $\cot 4\theta = 0$, $\cot^4\theta - 6\cot^2\theta + 1 = 0$

$\therefore z = \cot\frac{\pi}{8}, \cot\frac{3\pi}{8}$ are roots of $z^4 - 6z^2 + 1 = 0$ (1)

Sub. $z = -\cot\frac{\pi}{8}$ in: $(-\cot\frac{\pi}{8})^4 - 6(-\cot\frac{\pi}{8})^2 + 1 =$

$= \cot^4\frac{\pi}{8} - 6\cot^2\frac{\pi}{8} + 1 = 0$

Similarly for $-\cot\frac{3\pi}{8}$

Sub. $z = i\cot\frac{\pi}{8}$ in $z^4 + 6z^2 + 1$

$= (i\cot\frac{\pi}{8})^4 + 6(i\cot\frac{\pi}{8})^2 + 1$

Feedback

$= \cot^4\frac{\pi}{8} - 6\cot^2\frac{\pi}{8} + 1$

$= 0$ from (1)

Similarly for $\pm i\cot\frac{3\pi}{8}, -i\cot\frac{\pi}{8}$

$\therefore z = \pm i\cot\frac{\pi}{8}, \pm i\cot\frac{3\pi}{8}$ are solutions to $z^4 + 6z^2 + 1 = 0$.

10 c i) done well

ii) Sts struggled.
Most found $\frac{\pi}{8}$ or $\frac{3\pi}{8}$ & stopped there.
Most didn't know how to proceed to change the eq. into $z^4 + 6z^2 + 1$.

Solutions Q11)(a)(i)

Solutions to $z^7-1=0$ are $z=\text{cis}\frac{2k\pi}{7}$, $k=0,1,2,\ldots 6$

i.e. $z=1, \text{cis}\frac{2\pi}{7}, \text{cis}\frac{4\pi}{7}, \text{cis}\frac{6\pi}{7}, \text{cis}\frac{8\pi}{7}, \text{cis}\frac{10\pi}{7}, \text{cis}\frac{12\pi}{7}$. 2

[can also use principal argument].

Generally well done.

(ii) $w=\text{cis}\frac{2\pi}{7}$. $w^n=\text{cis}\frac{2n\pi}{7}$ [DeMoivre].

$[w^n]^7=[\text{cis}\frac{2n\pi}{7}]^7=\text{cis}\,2n\pi=1$. 1

$\therefore w^n$ will also be a root of $z^7-1=0$

Could also do $[w^n]^7=[w^7]^n=1^n=1$. Valid.

(iii) Roots of $z^7-1=0$ are $z=1, w, w^2, w^3, w^4, w^5, w^6$. (*)

By Sum of Roots of $z^7-1=0$, $1+w+w^2+w^3+w^4+w^5+w^6=0$ 1

$w+w^2+w^3+w^4+w^5+w^6=-1$.

SOR WAS penalized. → Don't use acronyms unless they are generally recognised [like congruency test].

(iv) $(w+w^6)(w^2+w^5)(w^3+w^4)$

$=(w^3+w^6+w^8+w^{11})(w^3+w^4)$

$=w^6+w^7+w^9+w^{10}+w^{11}+w^{12}+w^{14}+w^{15}$

$=w^6+1+w^2+w^3+w^4+w^5+1+w$ 2

$=2+w+w^2+w^3+w^4+w^5+w^6$ → Generally well done.

$=2-1$ [from (iii)]

$=1$.

(v) $w+w^6$

$=(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7})+(\cos\frac{-2\pi}{7}+i\sin\frac{-2\pi}{7})$

$=2\cos\frac{2\pi}{7}$ (1)

Similarly, $w^2+w^5=2\cos\frac{4\pi}{7}=-2\cos\frac{3\pi}{7}$ (2) ←

$w^3+w^4=2\cos\frac{6\pi}{7}=-2\cos\frac{\pi}{7}$ (3) ←

Generally well done, but some students didn't write & explain where $\cos\frac{\pi}{7}$ & $\cos\frac{3\pi}{7}$ came from (& lost a mark). 2

Hence $(w+w^6)(w^2+w^5)(w^3+w^4)$ $=1$ (from (iii))

$2\cos\frac{2\pi}{7}\times 2\cos\frac{3\pi}{7}\times -2\cos\frac{\pi}{7}$ $=1$

$8\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{\pi}{7}$ $=1$.

$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$ $=\frac{1}{8}$.

The MINIMUM was to write

$2\cos\frac{2\pi}{7}\times 2\cos\frac{4\pi}{7}\times 2\cos\frac{6\pi}{7}=1$

$2\cos\frac{2\pi}{7}\times -2\cos\frac{3\pi}{7}\times -2\cos\frac{\pi}{7}=1$

to get a mark.

Feedback

(*) Note with (iii): Another way was to note:

$(w-1)(w^6+w^5+w^4+w^3+w^2+w+1)=0$.

As $w\neq 1$, $w^6+w^5+w^4+w^3+w^2+w+1=0$.

This is valid (& was used).

↑ BUT YOU HAD TO WRITE $w\neq 1$.

Solutions Q.(11)(b)(i) Using sum of a Geometric Progression, ← Needed to state this.

$1+z+z^2+z^3+z^4+z^5+z^6 = \frac{z^7-1}{z-1}$ (*) → Could also use polynomial division.

Hence multiplying BS by $z-1$, $z^7-1=(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

(ii) ∴ From (a), roots of $z^6+z^5+z^4+z^3+z^2+z+1$

$= \text{cis}\frac{\pm 2\pi}{7}, \text{cis}\frac{\pm 4\pi}{7}, \text{cis}\frac{\pm 6\pi}{7}$.

$\therefore z^6+z^5+z^4+z^3+z^2+z+1 = \left(z-\text{cis}\frac{2\pi}{7}\right)\left(z-\text{cis}\left(-\frac{2\pi}{7}\right)\right)\left(z-\text{cis}\frac{4\pi}{7}\right)\left(z-\text{cis}\left(\frac{-4\pi}{7}\right)\right)$
$\left(z-\text{cis}\frac{6\pi}{7}\right)\left(z-\text{cis}\left(-\frac{6\pi}{7}\right)\right)$

This step HAD to be shown!

Noting that $\left(z-\text{cis}\frac{2\pi}{7}\right)\left(z-\text{cis}\left(-\frac{2\pi}{7}\right)\right)$
$=\left(z^2-z\left(\text{cis}\frac{2\pi}{7}+\text{cis}\left(-\frac{2\pi}{7}\right)\right)+1\right)$
$=\left(z^2-2z\cos\frac{2\pi}{7}+1\right)$

& similarly for $\left(z-\text{cis}\frac{4\pi}{7}\right)\left(z-\text{cis}\frac{-4\pi}{7}\right)$ & $\left(z-\text{cis}\frac{6\pi}{7}\right)\left(z-\text{cis}\frac{-6\pi}{7}\right)$

$z^6+z^5+z^4+z^3+z^2+z+1=\left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2-2z\cos\frac{4\pi}{7}+1\right)\left(z-2z\cos\frac{6\pi}{7}+1\right)$

Full marks to here

$=\left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2+2z\cos\frac{3\pi}{7}+1\right)\left(z^2+2z\cos\frac{\pi}{7}+1\right)$

(ii) Generally well done.

Feedback (*) Note: Some students expanded

$(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$
$=z^7+z^6+z^5+z^4+z^3+z^2+z-z^6-z^5-z^4-z^3-z^2-z-1$
$=z^7-1$. which is valid & got the mark.

If polynomial division was used, you HAD TO SHOW THE WHOLE DIVISION.

Solutions

Q.11(b)(iii)

From (ii)

$$z^6+1+z^5+z+z^4+z^2+z^3=\left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2+2z\cos\frac{3\pi}{7}+1\right)\left(z^2+2z\cos\frac{\pi}{7}+1\right)$$

(÷ z ÷ z ÷ z)

(iii) Generally well done.

Dividing BS by z^3

$$z^3+\frac{1}{z^3}+z^2+\frac{1}{z^2}+z+\frac{1}{z}+1=\left(z+\frac{1}{z}-2\cos\frac{2\pi}{7}\right)\left(z+\frac{1}{z}+2\cos\frac{3\pi}{7}\right)\left(z+\frac{1}{z}+2\cos\frac{\pi}{7}\right)$$

(iv) Letting $u=z+\frac{1}{z}$, $z^2+\frac{1}{z^2}=\left(z^2+2+\frac{1}{z^2}\right)-2$ (*)

$$=\left(z+\frac{1}{z}\right)^2-2$$

$$=u^2-2 \quad (1)$$

$$z^3+\frac{1}{z^3}=\left(z+\frac{1}{z}\right)\left(z^2-1+\frac{1}{z^2}\right)$$

$$=u\left(u^2-3\right)$$

$$=u^3-3u \quad (2)$$

Sub. $u=z+\frac{1}{z}$ in (iii)

$$u^3-3u+u^2-2+u+1=\left(u-2\cos\frac{2\pi}{7}\right)\left(u+2\cos\frac{3\pi}{7}\right)\left(u+2\cos\frac{\pi}{7}\right)$$

$$u^3+u^2-2u-1=\left(u-2\cos\frac{2\pi}{7}\right)\left(u+2\cos\frac{3\pi}{7}\right)\left(u+2\cos\frac{\pi}{7}\right)$$

Hence, $2\cos\frac{2\pi}{7}$, $-2\cos\frac{3\pi}{7}$ & $-2\cos\frac{\pi}{7}$ are the roots of $u^3+u^2-2u-1=0$.

END OF SOLUTIONS!

Feedback

Note for (iv)*

Some students expanded u^3+u^2-2u-1

$$=\left(z+\frac{1}{z}\right)^3+\left(z+\frac{1}{z}\right)^2-2\left(z+\frac{1}{z}\right)-1$$

$$=z^3+\frac{1}{z^3}+3z+\frac{3}{z}+z^2+\frac{1}{z^2}+2-2z-\frac{2}{z}-1$$

$$=z^3+\frac{1}{z^3}+z^2+\frac{1}{z^2}+z+\frac{1}{z}+1.$$

& this was also valid.